REMAINDER AND FACTOR THEOREM

A) Remainder Theorem: If a polynomial is divided by $(x-a)$, then the remainder is $f(a)$.

Example: If we divide the polynomial $f(x) = x^3 + 2x^2 - 5x + 6$ by $(x-2)$, then the remainder should be $f(2)$.

$$f(2) = 2^3 + 2 \cdot 2^2 - 5 \cdot 2 + 6 = 12$$

Proof:

$$\begin{array}{r|l} & x^2 + 4x + 3 \\ \hline x-2 & x^3 + 2x^2 - 5x + 6 \\ & \underline{x^3 - 2x^2} \\ & \quad 4x^2 - 5x \\ & \quad \underline{4x^2 - 8x} \\ & \qquad 3x + 6 \\ & \qquad \underline{3x - 6} \\ & \qquad\quad 12 \end{array}$$

B) Factor Theorem: If a polynomial is divided by $(x-a)$, and the remainder is zero, i.e. $f(a) = 0$, then $(x-a)$ is a factor of the polynomial.

Example: $f(x) = x^3 - 7x - 6$

Since $f(-2) = 0$, $f(3) = 0$, and $f(-1) = 0$

Then $(x+2)$, $(x-3)$, and $(x+1)$ are factors,

i.e $f(x) = x^3 - 7x - 6 = (x+2)(x-3)(x+1)$

Delta Ex 26.2 pg 239 Q 1-4, extension Q5 - 11

C) Factorising using the Factor Theorem

Example: Factorise $x^3 - 4x^2 + x + 6$

- The cubic expression has three roots which are all factors of the constant term in the expression, i.e. 6.
- The factors of 6 are 1, 2, 3, 6 and -1, -2, -3, -6. These are the possible numbers to substitute into the factor theorem process (may use the TABLE function of your graphics calculator to quickly work out which substitutions give you a value of zero).
  $f(1) = 6$
  $f(-1) = 0$ => $(x + 1)$ is a factor
  $f(2) = 0$ => $(x - 2)$ is a factor
- Therefore $x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - a)$, we can work out the final factor by inspection: $1 \times -2 \times -a$ must equal the constant term 6, so $a = 3$.
- Therefore $x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3)$

D) Solving polynomial equations using the Factor Theorem

Example: Solve the equation $3x^3 + 4x^2 - 5x - 2 = 0$

- Find one factor first using the factor theorem on numbers such as 1, -1, 2, -2 etc.
- f(1) = $3(1)^3 + 4(1)^2 - 5(1) - 2 = 0$ => $(x - 1)$ is a factor
- If we divide the polynomial $3x^3 + 4x^2 - 5x - 2$ by the factor $(x - 1)$, we will end up with a quadratic which can be factorised in the usual way.

$$
\begin{array}{r|l}
 & 3x^2 + 7x + 2 \\
\hline
x-1 & 3x^3 + 4x^2 - 5x - 2 \\
 & \underline{3x^3 - 3x^2} \\
 & \quad\quad 7x^2 - 5x \\
 & \quad\quad \underline{7x^2 - 7x} \\
 & \quad\quad\quad\quad 2x - 2 \\
 & \quad\quad\quad\quad \underline{2x - 2} \\
 & \quad\quad\quad\quad\quad 0
\end{array}
$$

- Factorise $3x^2 + 7x + 2 = (3x + 1)(x + 2)$
- So $3x^3 + 4x^2 - 5x - 2 = (3x + 1)(x + 2)(x - 1)$, and the solutions to the equation $3x^3 + 4x^2 - 5x - 2 = 0$ are $x = \frac{-1}{3}, -2, and\ 1$.

Delta Ex 26.3 pg 243, Q 4-13, 21, 22. Extension: the rest.